On the Reformulation of Semi-Generalized Homeomorphisms and Generalized Semi-Homeomorphisms

Aye Aye Thin
University of Computer Studies, Yangon
a2thin.ucsy@gmail.com

Abstract

There exist a number of problems relating to homeomorphisms and semi-homeomorphisms in generalized topological structures. In this paper, a study to reformulate the properties of semi-generalized homeomorphisms and generalized semi-homeomorphisms is presented. Moreover, the relationships between these two types of homeomorphisms are also investigated.

Keywords- homeomorphism, semi-generalized homeomorphisms, generalized semi-homeomorphisms

1. Introduction

Semi-open maps, generalizations of homeomorphisms, semi-homeomorhisms were firstly introduced by Biswas [1]. Semi-homeomorphisms was continued to study by Crossley and Hildebrand in [2]. As an analogy of Maki [3], Caldas and Dontchev [4] also introduced the Λs-sets (resp. Vs-sets) which are intersections of semi-open (resp. union of semi-closed) sets. In 2001, three classes of maps called generalized Λ_s open, generalized Λ_s^c -homeoorphisms and generalized ΛIs homeomorphisms due to [1, 2] were studied in [5].

2. Preliminaries

In this section some preliminary concepts, definitions and some properties are presented before we demonstrate the approach we adapted in this paper.

2.1 Semi-open and Semi-closed sets

Definition 1:

A subset A of a topological space (X,τ) is said to be semi-closed (briefly s-closed) if there exists a closed set F such that Int (F) ⊆ A ⊆ F.

A subset B of (X,τ) is called semi-open (briefly s-open) if its complement X\B is semi-closed in (X,τ).

Definition 2:

A subset A of a topological space (X,τ) is said to be semi-open (briefly s-open) if there exists an open set V such that V ⊆ A ⊆ cl V where cl V denotes the closure of V.

The complement of a semi-open set is called semi-closed. (briefly s-closed).

Proposition 1: The above definitions 1 and 2 are equivalent.

2.2 Generalized open and closed sets and Generalized Semi-open and Semi-closed Sets

Definition 3:

A subset A of X said to be generalized open (written as g-open) if F ⊆ int A whenever F ⊆ A and F is closed in X.

The complement of a generalized open set is called generalized closed (written as g-closed).

Definition 4:

A subset A of a space (X,τ) is said to be generalized semi-open (written shortly as gs-open) if F ⊆ sint A whenever F ⊆ A and F is closed in (X,τ).

The complement of a generalized semi-open set is called generalized semi-closed (written shortly as gs-closed).

Thus we have the following diagrams (Figure 1).

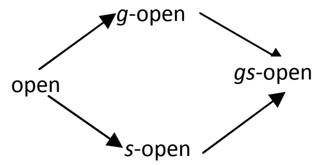

Figure 1 (a)

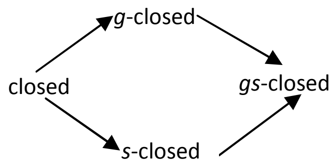

Figure 1 (b)

Example 1:

Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a\}, \{a, b\}, X\}$. Then

(i) $\{a, c\}$ is *s*-open, here *gs*-open, but not *g*-open.

(ii) $\{b\}$ is *g*-open, here *gs*-open, but not *s*-open.

Therefore, combining diagrams of Figure 1 and the above example, we have the diagrams in Figure 2.

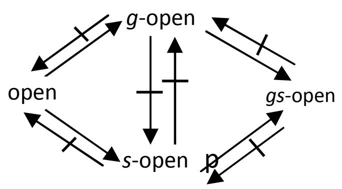

Figure 2 (a)

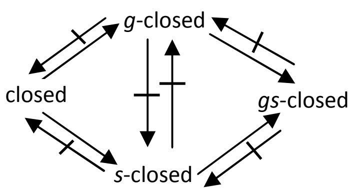

Figure 2 (b)

2.3 Semi-generalized open and Semi-generalized Closed sets

Definition 5:

A subset A is said to be semi-generalized closed set (written in short as sg-closed) if scl A ⊆ V whenever A ⊆ V and V is semi-open.

The complement of semi-generalized closed set is called semi-generalized open set (written in short as sg-open).

Proposition 2:

In a topological space, the followings are true.

(i) Every *s*-closed set is *sg*-closed . Consequently every *s*-open set is *sg*-open.

(ii) Every *sg*-closed set is *gs*-closed . Consequently every *sg*-open set is *gs*-open.

Consider the diagrams given in Figure 1(a), Figure 1(b) and Proposition 2, we have the diagrams as illustrated in Figure 3 and Figure 4.

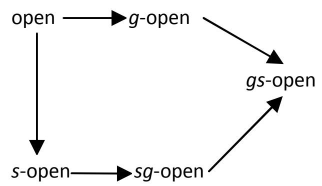

Figure 3 (a)

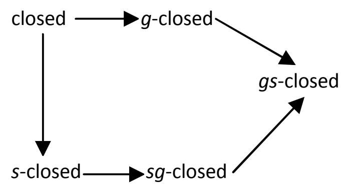

Figure 3 (b)

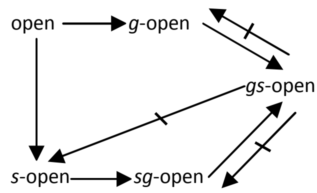

Figure 4 (a)

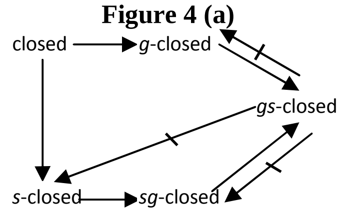

Figure 4 (b)

3. Reformulation of Semi-Generalized Homeomorphisms

In this section, the concepts of sg-homeomorphism, and sgc-homeomorphism are studied, the properties of these two types of homeomorphism are also reformulated.

The group structure of the set of all sgc-homeomorphisms is investigated, and the induced isomorphism between the group sgch (X,τ) and the group sgch (Y,σ) is obtained.

3.1 sg-Homeomorphisms

Definition 6:

A map f: (X,τ) → (Y, σ) is called irresolute if f^{-1} (V) is s-open in (X,τ) , for every s-open set V of (Y, σ).

Definition 7:

A map f: (X,τ) → (Y, σ) is called pre-semi-open if f(V) is s-open in (Y, σ), for every s-open set V of (X, τ).

Definition 8:

A map map f: (X,τ) → (Y, σ) is said to be semi-homeomorphism (B) (simply s.h. (B)) if

(i) f is bijective
(ii) f is continuous
(iii) f is s-open.

In this definition the letter (B) designates Biswas who first introduced this concept.

Definition 9:

A map f: (X,τ) → (Y, σ) is said to be semi-homeomorphism (C.H.) (simply s.h. (C.H.)) if

(i) f is bijective
(ii) f is irresolute
(iii) f is pre-semi-open.

In this definition the letter (C.H) designates Crossely and Hildebrand who first introduced this concept.

Definition 10:

A map f: (X,τ) → (Y, σ) is called a semi-generalized homeomorphism (abbreviated sg-homeomorphism) if

(i) f is bijective
(ii) f is sg-continuous
(iii) f is sg-open.

Proposition 3:

Every semi-homeomorphism (B) is an sg-homeomorphism

Proposition 4:

Every semi-homeomorphism (CH) is an sg-homeomorphism

Example 2:

$$X = Y = \{a, b, c\}$$

Define $\tau = \{\emptyset, \{a\}, X\}$ and $\sigma = \{\emptyset, \{b\}, \{a, b\}, Y\}$. Then $(X, \tau)$and $(Y, \sigma)$are topological spaces.

Define a map $f: (X, \tau) \to (Y, \sigma)$ by

$$f(a) = b$$
$$f(b) = a$$
$$f(c) = c.$$

Then f is an sg-homeomorphism from (X, τ) to (Y, σ). However, f is not a semi-homeomorphism (B).

Example 3:

$$X = Y = \{a, b, c\}$$

Define $\tau = \{\emptyset, \{a\}, \{b, c\}, X\}$ and $\sigma = \{\emptyset, \{a\}, \{b\}, \{a, b\}, Y\}$. Then (X, τ) and (Y,σ)are topological spaces.

Define a map $f: (X, \tau) \to (Y, \sigma)$ be the identity so that

$$f(a) = a$$
$$f(b) = b$$
$$f(c) = c.$$

Then f is an sg-homeomorphism from (X, τ) to (Y,σ). However, f is not a semi-homeomorphism (C.H.).

Remark 1:

We note that the identity map f: X → Y is not a semi-homeomorphism since it is not continuous. Thus, from examples 2 and 3, we have the diagram below.

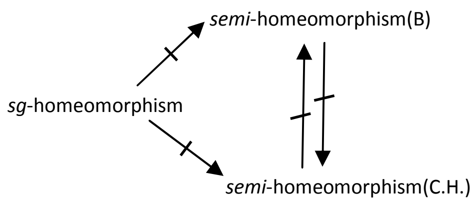

Figure 5.

3.2 sgc-homeomorphisms

Definition 11:

A map f: (X,τ) → (Y, σ) is called *sg*-irresolute if f^{-1} (F) is sg-closed in (X,τ), for every sg-closed set F in (Y, σ).

Definition 12:

A map f: (X,τ) →(Y, σ) is called an sgc-homeomorphism if

(1) f is bijective

(2) f is sg-irresolute

(3) The inverse map f^{-1} is sg-irresolute

Proposition 5:

Every semi-homeomorphism (C.H.) is an sgc-homeomorphism and every sgc-homeomorphism is an sg-homeomorhism.

Thus, semi-homeomorphism(CH)⇒sgc-homeomorphism⇒ sg-homeomorphism.

Proposition 6:

sgc-homeomorphism does not necessarily imply semi-homeomorphism (C.H).

4. Group structure of sgc-homeomorphisms

Theorem 1:

Let (X,τ) be a topological space. Then the followings are true;

(i) h (X,τ) ⊆ sh.B.⊆ sgh (X,τ)

(ii) h(X,τ) ⊆ sh.C.H. (X,τ) ⊆ sgch (X,τ) ⊆ sgh(X,τ).

(iii) Under composition multiplication, the set sgch (X,τ) is a group which contains sh.C.H. (X,τ) and h(X,τ) as subgroups.

Proposition 7:

It may happen that

(i) sh.C.H. (X,τ) is a proper subgroup of sgch (X,τ) .

In fact, an sgch will not necessarily be an sh.C.H.

(ii) sh.B. (X,τ) and sh.C.H. (X,τ) are proper subsets of sgh (X,τ) .

In fact, an sgh will not necessarily be an sh.C.H or an sh.B.

Proposition 8:

It may happen that sgch (X,τ) is a proper subset of sgh (X,τ).

In fact, an sgh will not necessarily be an sgch.

From remark 1, Propositions 6-8 and Theorem 1, we have the following Theorem 2.

Theorem 2:

If f: (X,τ) → (Y,σ) is a map, then we have the following diagram of implications:

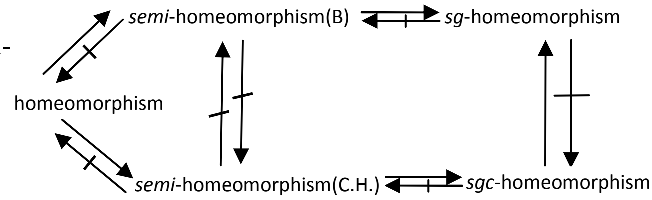

Figure 6.

where, p → q (resp. p ↛ q) represents that p implies q (resp. p does not necessarily imply q).

5. Generalized Semi-homeomorphisms

In this part, the concepts of gs-homemorphisms and gsc-homeomorphisms and their properties are studied. Then the group structure of the set of all gsc-homeomorphisms are also studied, and the induced isomorphism between the group gsch (X,τ) and gsch (Y,σ) is also presented.

5.1 gs-Homeomorphisms

Definition 13:

A map f: (X,τ) → (Y,σ) is called a generalized semi-homeomorphism (abbreviated gs-homeomorphism) if

(i) f is bijective

(ii) f is gs-continuous

(iii) f is gs-open

Proposition 9:

Every sg-homeomorphism is a gs-homeomorphism. But the converse is not true in general.

5.2 gsc-Homeomorphisms and gc-Homeomorphisms

Definition 14:

A map f: (X,τ) → (Y,σ) is called a gs-iresolute map if f^{-1}(F) is gs-closed in (X,τ) for every gs-closed set F is (Y,σ).

Definition 15:

A map f: (X,τ) →(Y,σ) is called a gsc-homeomorphism if

(i) f is bijective
(ii) f is gs-irresolute
(iii) The inverse map f^{-1} is gs-irresolute.

Proposition 10:
Every gsc-homeomorphism implies a gs-homeomorphism.

Definition 16 :
A map f: (X,τ) → (Y,σ) is called gc-irresolute if f^{-1} (F) is g-closed in (X,τ) for every g-closed set F in (Y,σ).

Definition 17 :
A map f: (X,τ) → (Y,σ) is called a gc-homeomorphism if
(i) f is bijective
(ii) f is gc-irresolute
(iii) The inverse map f^{-1} is gc-irresolute.

Example 4:
$X = Y = \{a, b, c\}$.
Define $\tau = \{\phi, \{a, b\}, X\}$ and
$\sigma = \{\phi, \{b\}, \{a, b\}, Y\}$. Then (X, τ) and (Y,σ)are topological spaces.
Define a map $f: (X, \tau) \to (Y, \sigma)$ be the identity map so that

$$f(a) = a$$
$$f(b) = b$$
$$f(c) = c.$$

Then f is a gc-homeomorphism from (X, τ) to (Y, σ). However, f is not a *gsc*- homeomorphism.

Example 5:
$X = Y = \{a, b, c\}$.
Define $\tau = \{\phi, \{a\}\{a, b\}, X\}$ and $\sigma = \{\phi, \{a\}, \{b\}, \{a, b\}, \{b, c\}, Y\}$. Then (X, τ) and (Y,σ)are topological spaces.
Define a map $f: (X, \tau) \to (Y, \sigma)$ by

$$f(a) = b$$
$$f(b) = a$$
$$f(c) = c.$$

Then f is a gsc-homeomorphism from (X, τ) to (Y, σ). However, f is not a *gc*-homeomorphism.

Remark 2 :
It can be shown that the concepts of gsc-homeomorphisms and gc-homeomorphisms are independent of each other using examples 4, 5.

Corollary 1:
Every homeomorphism is a gsc-homeomorphism.

The converse is not true is general as we shall see in Example 7.

Remark 3:
The following two examples show that the concepts of semi-homeomorphisms (C.H.) and gsc-homeomorphisms are independent of each other.

Example 6:

$$X = \{a, b, c\}$$

Define $\tau = \{\emptyset, \{a\}, \{a, b\}, X\}$ .Then (X, τ) is a topological space.
Define a map $f: (X, \tau) \to (X, \tau)$ by

$$f(a) = c$$
$$f(b) = b$$
$$f(c) = a.$$

Then f is a semi-homeomorphism (C.H.) .
However, f is not a *gsc*- homeomorphism.

Example 7:

$$X = Y = \{a, b, c\}$$

Define $\tau = \{\emptyset, \{a\}, \{a, b\}, X\}$ and $\sigma = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c\}, Y\}$. Then (X, τ) and (Y,σ)are topological spaces.
Define a map $f: (X, \tau) \to (Y, \sigma)$ by

$$f(a) = b$$
$$f(b) = \text{a}$$
$$f(c) = c.$$

Then f is not a semi-homeomorphism (C.H.) and so it is not a homeomorphism. However, f is a *gsc*- homeomorphism.

5.3 g.Homeomorphisms

Definition 18:
A map f: (X,τ) → (Y,σ) is called a g-homeomorphism if
(i) f is bijective
(ii) f is g-continuous
(iii) f is g-open

Proposition 11:
Every g-homeomorphism is a gs-homeomorphism.

Remark 4:
From corollaries, examples, theories and remarks described in previous sections, Figure 7 is obtained.

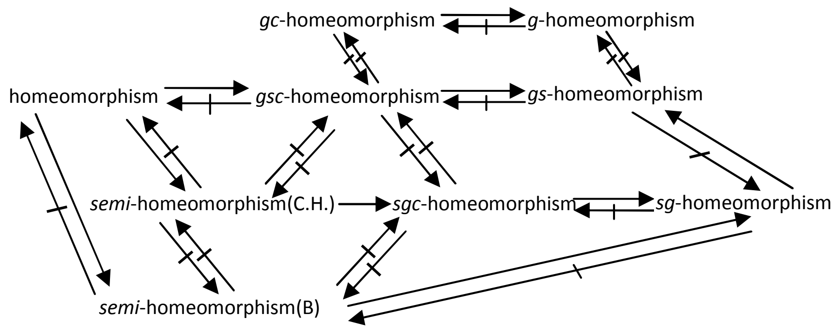

Figure 7

where p→ $\boldsymbol{q}(\boldsymbol{resp}, \boldsymbol{p} \to \boldsymbol{q})$represents that p implies q(resp.p does not always imply q).

6. Computing Homeomorphism

Digitization is not as easy as it looks. If one digitizes a 3D object even with a dense sampling grid, the reconstructed digital object may have topological distortions. In [6], it was proved that it is possible to repair the obtained digital image in a locally bounded way so that it is homeomorphic and close to the 3D object. This study showed that the resulting digital object is always well-composed, which has nice implications for a lot of image analysis problems. Two algorithms for constructions of homeomorphic sub-trees were proposed in [7] to solve the problems in computational evolutionary biology.

7. Conclusion

In this paper, a number of problems relating to homeomorphisms and semi-homeomorphisms in generalized topological structures is presented. Three topological structures a generalized topological structures sg-homeomorphisms and sgc-homeomorphisms certain problems in relation to gs-homeomorphims, gsc-homeomorphisms, gc-homeomorphisms and g-homeomorphisms are studied.

8. References

[1] N. Biswas, “On Some Mappings in Topological Spaces,” *Calcutta Math. Soc.* vol: 61, pp. 127-135, 1969.

[2] G. Crossley and S. K. Hidebrand, “Semi-Topological Properties,” *Fund. Math.*, vol: 74, pp. 233-254, 1972.

[3] H. Maki, “Generalized Λ^c_s -sets and the Associeted Closure Operator”, *Special Issue in Commemoration of Prof. Kazusada IKEDA’s Retirement*, pp. 139-146, 1986.

[4] M. Caldas and J. Dontchev, “G. Λ_s-sets and G:Vs-sets”, Mem. Fac. Sci. Kochi Univ. (Math.), vol: 21, pp. 21-30, 2000.

[5] M. Caldas, “More on Generalized Homeomorphisms in Topological Spaces”, *Divulgaciones Matem´aticas* vol. 9, no. 1, pp. 55–63, 2001.

[6] P. Stelldinger, L. J. Jatecki and M. Siqueirs, ”Topological Equivalence between a 3D Object and the Reconstruction of its Digital Image,” *IEEE Trans. Pattern Anal. Mach. Intell.* 29(1): 126-140, 2007.

[7] M. R. Henzinger, V, King, and T. Warnow, “Constructing a Tree from Homeomorphic Subtrees, with Applications to Computational Evolutionary Biology”, *Proc. 7th Symp. Discrete Algorithms, ACM and SIAM*, pp. 333-340, 1996.